Solutions to Question Sheet 1, Limits I. v1. 2019-20

1. When verifying the ε - δ definition of $\lim_{x\to a} f(x) = L$ you need to know the value of the limit, L, in advance. This question is about finding L. Without detailed proofs evaluate the following limits.

i) $\lim\limits_{x\to 1} \dfrac{x^2 - x - 2}{x+1}$ ii) $\lim\limits_{x\to -1} \dfrac{x^2 - x - 2}{x+1}$

iii) $\lim\limits_{x\to 1} \left\{\dfrac{1}{x-1} - \dfrac{2}{x^2-1}\right\}$ iv) $\lim\limits_{t\to 9} \dfrac{9-t}{3-\sqrt{t}}$.

v) $\lim\limits_{x\to 2} \dfrac{\frac{1}{x} - \frac{1}{2}}{x-2}$. vi) $\lim\limits_{t\to 8} \dfrac{8-t}{2-\sqrt[3]{t}}$.

Hint: In part (iv) use the important identity

$$a - b = \left(\sqrt{a} - \sqrt{b}\right)\left(\sqrt{a} + \sqrt{b}\right)$$

for all $a, b \geq 0$. This follows from the "**difference of squares**" formula

$$x^2 - y^2 = (x - y)(x + y)$$

with $a = x^2$ and $b = y^2$.

For part (vi) use a similar result based on

$$x^3 - y^3 = (x - y)\left(x^2 + xy + y^2\right).$$

Solution i)

$$\lim_{x\to 1} \frac{x^2 - x - 2}{x+1} = \lim_{x\to 1} \frac{(x+1)(x-2)}{x+1} = \lim_{x\to 1} (x-2) = -1.$$

ii)

$$\lim_{x\to -1} \frac{x^2 - x - 2}{x+1} = \lim_{x\to -1} \frac{(x+1)(x-2)}{x+1} = \lim_{x\to -1} (x-2) = -3.$$

iii) Use partial fractions to write

$$\frac{2}{x^2-1}=\frac{1}{x-1}-\frac{1}{x+1}.$$

Then

$$\lim_{x\to 1}\left\{\frac{1}{x-1}-\frac{2}{x^2-1}\right\}=\lim_{x\to 1}\frac{1}{x+1}=\frac{1}{2}.$$

iv) Use the Hint to write $9-t=\left(3-\sqrt{t}\right)\left(3+\sqrt{t}\right)$. Then

$$\lim_{t\to 9}\frac{9-t}{3-\sqrt{t}}=\lim_{t\to 9}\frac{\left(3-\sqrt{t}\right)\left(3+\sqrt{t}\right)}{3-\sqrt{t}}=\lim_{t\to 9}\left(3+\sqrt{t}\right)=6.$$

v) For all $x\neq 2$ we have

$$\frac{\frac{1}{2}-\frac{1}{2}}{x-2}=\frac{0}{x-2}=0,$$

so

$$\lim_{x\to 2}\frac{\frac{1}{2}-\frac{1}{2}}{x-2}=0.$$

vi) Use

$$a-b=\left(a^{1/3}-b^{1/3}\right)\left(a^{2/3}+a^{1/3}b^{1/3}+b^{2/3}\right)$$

with $a=8$ and $b=t$ to get

$$\lim_{t\to 8}\frac{8-t}{2-\sqrt[3]{t}}=\lim_{t\to 8}\left(4+2t^{1/3}+t^{2/3}\right)=12.$$

2. Consider the following **Rough Work** when trying to verify the ε-δ definition of $\lim_{x\to 2}x^2=4$.

Assume $0<|x-2|<\delta$. Consider

$$|f(x)-L|=\left|x^2-4\right|=|(x-2)(x+2)|<\delta\,|x+2|.$$

Assume $\delta\leq 1$ so $0<|x-2|<\delta\leq 1$, i.e. $-1<x-2<1$ and thus $3<x+2<5$. For then

$$\left|x^2-4\right|<\delta\,|x+2|<5\delta,$$

which we want $\leq \varepsilon$. Hence choose $\delta = \min(1, \varepsilon/5)$.

Question What do we get for δ if we replace the requirement $\delta \leq 1$ by

$$\text{i) } \delta \leq 100 \quad \text{or} \quad \text{ii) } \delta \leq 1/100\,?$$

Solution i) If $\delta \leq 100$ then $0 < |x-2| < \delta \leq 100$. This implies $|x+2| \leq 104$ and so the choice of δ could be

$$\delta = \min\left(100, \frac{\varepsilon}{104}\right).$$

ii) If $\delta \leq 1/100$ then $0 < |x-2| < \delta \leq 1/100$. This implies $|x+2| \leq 401/100$ and so the choice of δ could be

$$\delta = \min\left(\frac{1}{100}, \frac{100}{401}\varepsilon\right).$$

Limits of Cubic Polynomials

In the next four questions we look at limits of cubic polynomials. There are so many questions because I want to highlight different aspects of the quadratic polynomial which arises.

3. i) Factorise $x^3 - 8$ into a linear and a quadratic factor.

   ii) Bound, from above,
$$\left|x^2 + 2x + 4\right|$$
   on the interval $1 < x < 3$.

   iii) Show that the ε - δ definition of
$$\lim_{x\to 2} x^3 = 8,$$
   is satisfied if we choose $\delta = \min(1, \varepsilon/19)$ given $\varepsilon > 0$.

   Solution i) You should recognise $x^3 - 8 = x^3 - 2^3$ from the Hint in Question 1. From there we get the factorization
$$x^3 - 8 = (x-2)\left(x^2 + 2x + 4\right).$$

ii) On $1 < x < 3$ we have $1 < x^2 < 9$ and so

$$7 = 1 + 2 + 4 < x^2 + 2x + 4 < 9 + 6 + 4 = 19,$$

in which case $|x^2 + 2x + 4| < 19$.

iii) Let $\varepsilon > 0$ be given. Choose $\delta = \min(1, \varepsilon/19)$. This means both $\delta \le 1$ and $\delta \le \varepsilon/19$. Assume $0 < |x-2| < \delta$, then

$$\begin{aligned} |f(x) - L| &= \left|x^3 - 8\right| \\ &= |x-2|\left|x^2 + 2x + 4\right| \\ &< \delta 19 \quad \text{using } |x-2| < \delta \text{ and part ii}, \\ &< \left(\frac{\varepsilon}{19}\right) 19 = \varepsilon. \end{aligned}$$

Hence we have verified the ε - δ definition of limit.

4. Given $\varepsilon > 0$ find a $\delta > 0$ that verifies the ε - δ definition of

$$\lim_{x \to 3} x^3 = 27.$$

Solution This time

$$f(x) - L = x^3 - 27 = (x-3)\left(x^2 + 3x + 9\right).$$

If $\delta \le 1$ then $0 < |x-3| < \delta \le 1$ which opens out as $2 < x < 4$. For such x, we have $4 < x^2 < 16$. Thus

$$19 = 4 + 6 + 9 < x^2 + 3x + 9 < 16 + 12 + 9 = 37.$$

Hence $|x^2 + 3x + 9| < 37$ and we can choose $\delta = \min(1, \varepsilon/37)$.

5. i) Factorise $x^3 - 6x - 4$.

ii) Bound, from above, $|x^2 - 2x - 2|$ on the interval $|x+2| < 1$.

iii) Verify the ε - δ definition of

$$\lim_{x \to -2} \left(x^3 - 6x - 2\right) = 2,$$

i.e. given $\varepsilon > 0$ find a $\delta > 0$ for which the definition is satisfied.

Solution i) Test small integers x to find a root of $x^3 - 6x - 4$. We find $x = -2$ is a root, so $x + 2$ is a factor. Then, for example by equating coefficients in

$$x^3 - 6x - 4 = (x+2)\left(ax^2 + bx + c\right),$$

we find that

$$x^3 - 6x - 4 = (x+2)\left(x^2 - 2x - 2\right).$$

ii) The interval $|x+2| < 1$ is $-3 < x < -1$.

Careful! Without thinking you might assume that a quadratic with positive leading term attains it's maximum value at the right hand end of an interval, which it does in Questions 3 and 4. But in the present case the quadratic $x^2 - 2x - 2$ equals 13 at $x = -3$, and -2 at $x = 0$. So the maximum is **not** at the right hand end of the interval.

But, we only require **an** upper bound on $|x^2 - 2x - 2|$, **not** the best. A *recommended* method is to use the triangle inequality

$$\left|x^2 - 2x - 2\right| \leq \left|x^2\right| + |-2x| + |-2| = |x|^2 + 2|x| + 2.$$

Also $-3 < x < -1$ implies $|x| < 3$, which together gives

$$\left|x^2 - 2x - 2\right| < 3^2 + 2 \times 3 + 2 = 17.$$

iii) Let $\varepsilon > 0$ be given. Choose $\delta = \min(1, \varepsilon/17)$. Assume that $0 < |x+2| < \delta$. Then

$$\begin{aligned} |f(x) - L| &= \left|\left(x^3 - 6x - 2\right) - 2\right| \\ &= \left|x^3 - 6x - 4\right| \\ &= \left|(x+2)\left(x^2 - 2x - 2\right)\right| \\ &< \delta\left|x^2 - 2x - 2\right| \\ &\leq 17\delta \end{aligned}$$

by the argument above, allowable since $|x+2| < \delta \leq 1$. Next using $\delta \leq \varepsilon/17$ gives

$$\left|x^3 - 6x - 4\right| \leq 17(\varepsilon/17) = \varepsilon,$$

as required to verify the ε - δ definition.

Note If we had been asked to find the *least* upper bound of $|x^2 - 2x - 2|$ on $-3 < x < -1$ we might start by completing the square

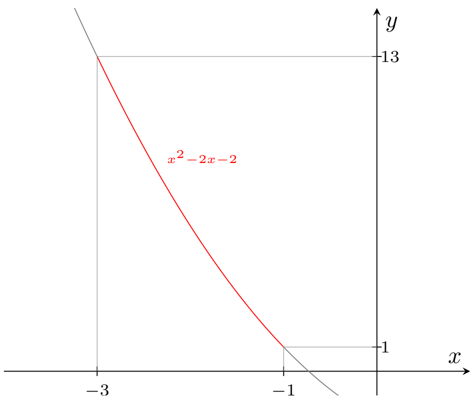
$$x^2 - 2x - 2 = (x-1)^2 - 3.$$

Then

$$\begin{aligned} -3 < x < -1 &\implies -4 < x - 1 < -2 \\ &\implies 4 < (x-1)^2 < 16 \\ &\implies 1 < (x-1)^2 - 3 < 13 \\ &\implies \left|(x-1)^2 - 3\right| < 13. \end{aligned}$$

Thus 13 is the smallest upper bound. The ε - δ definition of limit would then be verified with $\delta = \min(1, \varepsilon/13)$.

Graphically, $x^2 - 2x - 2$ on $[-3, -1]$ is



End of Note.

6. i) Factorise $x^3 - 4x^2 + 4x - 1$.

   ii) Bound from above $|x^2 - 3x + 1|$ on the interval $0 < |x - 1| < 1$.

   iii) Verify the ε - δ definition of

   $$\lim_{x\to 1} \left(x^3 - 4x^2 + 4x + 1\right) = 2,$$

   i.e. given $\varepsilon > 0$ find a $\delta > 0$ for which the definition is satisfied.

Solution i) A search through small integers will quickly find the root $x = 1$. There is thus a factor of $x - 1$ and, by equating coefficients,

$$x^3 - 4x^2 + 4x - 1 = (x - 1)\left(x^2 - 3x + 1\right).$$

ii) The last question showed that a quadratic with positive leading term might not be maximal at the right hand end of an interval. In question 5 the maximum was attained at the left hand end. In the present example, on the interval $0 < |x - 1| < 1$, i.e. $0 < x < 2$ the quadratic factor $|x^2 - 3x + 1|$ is **not** maximal at either end point. But again we simply use the triangle inequality

$$\left|x^2 - 3x + 1\right| \leq |x|^2 + 3|x| + 1 < 2^2 + 3 \times 2 + 1 = 11.$$

Thus the ε - δ definition of limit is verified on the choice of $\delta = \min(1, \varepsilon/11)$.

iii) Let $\varepsilon > 0$ be given. Choose $\delta = \min(1, \varepsilon/11)$. Assume that $0 < |x - 1| < \delta$. Then

$$\begin{aligned} |f(x) - L| &= \left|\left(x^3 - 4x^2 + 4x + 1\right) - 2\right| \\ &= \left|x^3 - 4x^2 + 4x - 1\right| \\ &= \left|(x - 1)\left(x^2 - 3x + 1\right)\right| \\ &< \delta\left|x^2 - 3x + 1\right| \\ &\leq 11\delta, \end{aligned}$$

by the argument above, allowable since $|x - 1| < \delta \leq 1$. Next using $\delta \leq 4\varepsilon/5$ gives

$$\left|x^3 - 6x - 4\right| \leq 11(\varepsilon/11) = \varepsilon,$$

as required to verify the ε - δ definition.

Note The upper bound of 11 on the quadratic factor is quite poor. The alternative method is to complete the squares so

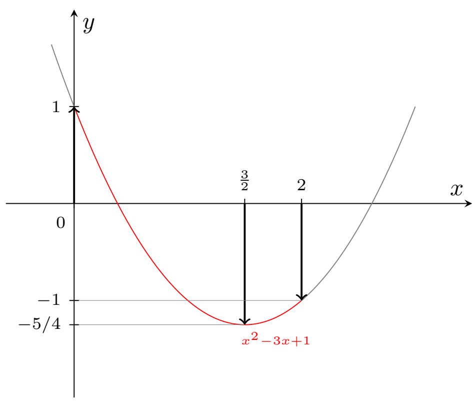
$$x^2 - 3x + 1 = \left(x - \frac{3}{2}\right)^2 - \frac{5}{4}.$$

Then

$$\begin{aligned} 0 < x < 2 \quad &\implies \quad -\frac{3}{2} < x - \frac{3}{2} < \frac{1}{2} \\ &\implies \quad 0 \le \left(x - \frac{3}{2}\right)^2 < \frac{9}{4} \\ &\implies \quad -\frac{5}{4} \le \left(x - \frac{3}{2}\right)^2 - \frac{5}{4} < 1 \\ &\implies \quad \left|\left(x - \frac{3}{2}\right)^2 - \frac{5}{4}\right| \le \frac{5}{4}, \end{aligned}$$

with equality at $x = 3/2$. Thus the maximum occurs at the turning point of the quadratic. Make sure you understand each implication in this chain. This much improved bound would allow us to choose $\delta = \min(1, 4\varepsilon/5)$.

Graphically, $x^2 - 3x + 1$ on $[0, 2]$ is



End of Note.

Limits of Rational Functions

In the next two questions we take a result $\lim_{x\to a} f(x) = L$ and examine

$$\lim_{x\to a} \frac{f(x) - L}{x - a},$$

for this gives examples of limits of rational functions which are **not** defined at the limit point.

7. (Based on Question 3.iii). i) Calculate, without proof,

$$\lim_{x\to 2}\frac{x^3-8}{x-2}.$$

ii) Fully factorise the polynomial

$$x^3-12x+16.$$

iii) Prove the value found in Part i is correct by verifying the ε - δ definition of limit.

Solution i) Recall $a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$ which leads to

$$\lim_{x\to 2}\frac{x^3-8}{x-2}=\lim_{x\to 2}\frac{(x-2)\left(x^2+2x+4\right)}{x-2}=\lim_{x\to 2}\left(x^2+2x+4\right)=12,$$

without justification.

ii) A small integer root of $x=2$ is quickly found which leads to

$$\begin{aligned}x^3-12x+16 &= (x-2)\left(x^2+2x-8\right)\\ &= (x-2)^2(x+4)\,.\end{aligned}$$

iii) To verify the ε - δ definition consider

$$\begin{aligned}|f(x)-L| &= \left|\frac{x^3-8}{x-2}-12\right|\\ &= \left|\frac{x^3-12x+16}{x-2}\right|\\ &= \left|\frac{(x-2)^2(x+4)}{x-2}\right| \quad\text{by part ii,}\\ &= |x-2|\,|x+4|\,.\end{aligned}$$

Let $\varepsilon > 0$ be given. Choose $\delta = \min(1, \varepsilon/7)$. This means both $\delta \leq 1$ and $\delta \leq \varepsilon/7$. Assume $0 < |x-2| < \delta \leq 1$ which implies $1 < x < 3$ in which case $|x+4| < 7$. Then

$$\begin{aligned} |f(x) - L| &= |x-2|\,|x+4| \\ &< \delta 7 \leq \left(\frac{\varepsilon}{7}\right) 7 \\ &= \varepsilon. \end{aligned}$$

Hence we have verified the ε - δ definition of

$$\lim_{x\to 2} \frac{x^3-8}{x-2} = 12.$$

8. (Based on Question 5.) i) What is the value of

$$\lim_{x\to -2} \frac{x^3-6x-4}{x+2}\ ?$$

ii) Prove your result by verifying the ε - δ definition of this limit.

Solution i) We have already discovered in Question 5 that

$$x^3 - 6x - 4 = (x+2)\left(x^2 - 2x - 2\right).$$

So

$$\begin{aligned} \lim_{x\to -2} \frac{x^3-6x-4}{x+2} &= \lim_{x\to -2} \frac{(x+2)(x^2-2x-2)}{x+2} \\ &= \lim_{x\to -2} \left(x^2 - 2x - 2\right) = 6, \end{aligned}$$

without justification.

ii) Let $\varepsilon > 0$ be given. Choose $\delta = \min(1, \varepsilon/7)$. This means both $\delta \leq 1$ and $\delta \leq \varepsilon/7$.

Assume $0 < |x-(-2)| < \delta \leq 1$ which implies $-3 < x < -1$. Subtracting 4 throughout gives $-7 < x - 4 < -5$ which implies $|x-4| < 7$.

Then

$$
\begin{aligned}
|f(x) - L| &= \left| \frac{x^3 - 6x - 4}{x+2} - 6 \right| \\
&= \left| \frac{x^3 - 12x - 16}{x+2} \right| \\
&= \left| \frac{(x-4)(x+2)^2}{x+2} \right| \quad \text{by part i,} \\
&= |x-4|\,|x+2| \\
&< 7\delta \quad \text{using } |x+2| < \delta \text{ and } |x-4| < 7, \\
&\le 7\left(\frac{\varepsilon}{7}\right) = \varepsilon.
\end{aligned}
$$

Hence we have verified the ε - δ definition of

$$\lim_{x \to -2} \frac{x^3 - 6x - 4}{x+2} = 6.$$

In the previous two questions we have looked at the limits of rational functions at a point where the function is **not** defined. Now we look at examples where the rational function **is** well-defined at the limit point.

9. i) Show that

$$\frac{3}{4} < \frac{x+2}{x+3} < \frac{5}{6}$$

for $1 < x < 3$.

ii) Show that the ε - δ definition of

$$\lim_{x \to 2} \frac{x^2 + 2x + 2}{x+3} = 2$$

can be verified by the choice of $\delta = \min(1, 6\varepsilon/5)$.

Solution i. Since $x > 1$ we have $x + 3 > 0$ and we can multiply up without changing the direction of any inequality. Thus

$$\begin{aligned} \frac{3}{4} < \frac{x+2}{x+3} < \frac{5}{6} &\iff 18\,(x+3) < 24\,(x+2) < 20\,(x+3) \\ &\iff 6 < 6x < 2x + 12, \end{aligned}$$

having subtracted $18x + 48$ from all sides. Then $6 < 6x < 2x + 12$ iff $1 < x$ and $4x < 12$, i.e. $1 < x < 3$.

ii. Let $\varepsilon > 0$ be given. Choose $\delta = \min(1, 6\varepsilon/5)$. This means both $\delta \leq 1$ and $\delta \leq 6\varepsilon/5$.

Assume $0 < |x-2| < \delta \leq 1$ which expands as $1 < x < 3$. For such x we have $|(x+2)/(x+3)| < 5/6$ by part i. Then

$$\begin{aligned} |f(x) - L| &= \left| \frac{x^2 + 2x + 2}{x+3} - 2 \right| = \left| \frac{x^2 - 4}{x+3} \right| = |x-2| \left| \frac{x+2}{x+3} \right| \\ &< \delta \frac{5}{6} \leq \left(\frac{6}{5} \varepsilon \right) \frac{5}{6} = \varepsilon. \end{aligned}$$

Hence we have verified the ε - δ definition of

$$\lim_{x \to 2} \frac{x^2 + 2x + 2}{x+3} = 2.$$

10. Evaluate

$$\lim_{x \to 2} \frac{x^2 - 2x - 12}{x+2}$$

and verify the ε - δ definition of the limit.

Solution We might guess that the limit is

$$\lim_{x \to 2} \frac{x^2 - 2x - 12}{x+2} = \frac{-12}{4} = -3.$$

Let $\varepsilon > 0$ be given. Choose $\delta = \min(1, 3\varepsilon/4)$. This means both $\delta \leq 1$ and $\delta \leq 3\varepsilon/4$.

Assume $0 < |x-2| < \delta \le 1$ which implies $1 < x < 3$. For such x we have

$$\left|\frac{x+3}{x+2}\right| < \frac{4}{3},$$

proved by inverting the *lower* inequality in Part i of the previous question. Then

$$\begin{aligned} |f(x)-L| &= \left|\frac{x^2-2x-12}{x+2} - (-3)\right| \\ &= \left|\frac{x^2+x-6}{x+2}\right| \\ &= |x-2|\left|\frac{x+3}{x+2}\right| \\ &< \delta\frac{4}{3} \le \left(\frac{3}{4}\varepsilon\right)\frac{4}{3} = \varepsilon. \end{aligned}$$

Hence we have verified the ε - δ definition of

$$\lim_{x\to 2} \frac{x^2-2x-12}{x+2} = -3.$$

Finally

11. Why must any $\delta > 0$ used to verify the ε - δ definition of the limit of $\sqrt{x}$ as $x \to 9$ satisfy $\delta \le 9$?

    Given $\varepsilon > 0$ find a $\delta > 0$ for which the definition of

    $$\lim_{x\to 9} \sqrt{x} = 3$$

    is satisfied.

    Hint Use the Hint to Question 1.

    Solution To verify $\lim_{x\to 9}\sqrt{x} = 3$ we need look at $x : 0 < |x-9| < \delta$, i.e.

    $$9 - \delta < x < 9 + \delta.$$

For $\sqrt{x}$ to be defined we need $x > 0$ and thus we require $9 - \delta \geq 0$, i.e. $\delta \leq 9$.

Rough Work. The hint given in the question refers to the identity

$$\left(\sqrt{a} - \sqrt{b}\right)\left(\sqrt{a} + \sqrt{b}\right) = a - b,$$

for non-negative a and b. We use this by multiplying $\sqrt{x} - 3$ by 1 in the form

$$1 = \frac{\sqrt{x} + 3}{\sqrt{x} + 3}.$$

For then

$$\left|\sqrt{x} - 3\right| = \left|\frac{(\sqrt{x} + 3)}{(\sqrt{x} + 3)}\left(\sqrt{x} - 3\right)\right| = \left|\frac{x - 9}{\sqrt{x} + 3}\right|. \tag{1}$$

It is possible to demand $\delta \leq 1$ in which case $8 < x < 10$ and then

$$\frac{1}{\sqrt{x} + 3} < \frac{1}{\sqrt{8} + 3}.$$

It would be possible to choose

$$\delta = \min\left(1, \left(\sqrt{8} + 3\right)\varepsilon\right).$$

We saw above that we must have $\delta < 9$. For this we find $0 < x < 18$, in which case $1/(\sqrt{x} + 3) < 1/3$. Then we can choose $\delta = \min(9, 3\varepsilon)$.

My preferred option is $\delta < 9$ along with the simple upper bound of $1/(\sqrt{x} + 3) < 1$. Then we can choose $\delta = \min(9, \varepsilon)$.

What we see here is an example of the fact that if the ε - δ definition is satisfied for **an** $\varepsilon > 0$ then it is satisfied by **all** $\varepsilon' < \varepsilon$.

End of Rough Work.

Solution Let $\varepsilon > 0$ be given. Choose $\delta = \min(9, \varepsilon)$. Assume $0 < |x - 9| < \delta$.

First $0 < |x - 9| < \delta \leq 9$ implies $x > 0$ in which case $\sqrt{x}$ is defined. Then, using $\sqrt{x} + 3 \geq 3 > 1$, and $\delta \leq \varepsilon$, we have

$$|f(x) - L| = \left|\sqrt{x} - 3\right| = \left|\frac{x - 9}{\sqrt{x} + 3}\right| < \frac{\delta}{1} \leq \varepsilon$$

as required. Thus we have verified the ε - δ definition of $\lim_{x\to 9}\sqrt{x} = 3$.